Mathematical methods in physics - Lagrange and Hamilton mechanics

Homework No. 1, 04.12.2023

Deadline: 18.11.2023, 09:20
where: during the lecture or mail to klawe@math.uni-heidelberg.de
Definition. The unite vector:

$$\vec{b} = \vec{t} \times \vec{n}, \tag{0.1}$$

is called a binormal unite vector. The triple of vectors $(\vec{t}, \vec{n}, \vec{b})$ is orthonormal. ($\vec{t}$ is a unite tangent and $\vec{n}$ is a unite normal vector.)
Theorem. (Frenet) Let $s \to \vec{x}(s) = (x_1(s), x_2(s), x_3(s))^T$ be a curve in $\mathbb{R}^3$ endowed with the natural parametrisation. Then the following equations hold:

$$\begin{aligned} \frac{d\vec{t}}{ds} &= \kappa(s)\vec{n}(s), \\ \frac{d\vec{n}}{ds} &= -\kappa(s)\vec{t}(s) - \chi(s)\vec{b}(s), \\ \frac{d\vec{b}}{ds} &= \chi(s)\vec{n}(s), \end{aligned} \tag{0.2}$$

where $\chi(s)$ is called the torsion (or second curvature) of the curve.
1. Prove Frenet theorem for curve in $\mathbb{R}^3$.
2. A point particle of unite mass is moving along the x-axis under the action of a conservative force with potential

$$V(x) = \begin{cases} (x+1)^2, & \text{if } x \leq -1, \\ 0, & \text{if } -1 < x < -1, \\ (x-1)^2, & \text{if } 1 < x. \end{cases} \tag{0.3}$$

Draw the phase curve corresponding to the values $E = 0, \frac{1}{2}, 1$. Prove that the period T of the motion corresponding to a fixed value $E > 0$ of the system energy is

$$T = 2\pi \left(\frac{1}{\sqrt{2}} + \frac{1}{\pi} \sqrt{\frac{2}{E}} \right). \tag{0.4}$$

3. Consider the spiral of Archimedes:

$$\begin{aligned} x_1(t) &= rt\cos t, \\ x_2(t) &= rt\sin t, \end{aligned} \tag{0.5}$$

and compute the velocity, acceleration, natural parametrisation, unite normal and tangent vector and curvature.